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MATTER IN FOUR DIMENSIONS INDUCED BY GEOMETRY OF FIVE DIMENSIONS

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ABSTRACT

In this paper induced matter theory is studied. It is shown how matter in 4D can be interpreted as a manifestation of 5D geometry. A new solution is presented which generalizes the well known Ponce de Leon solution. Some properties of the new solution are discussed.

KEYWORDS: Kaluza-Klein Theory, Extra Dimension, Induced Matter Theory, Ricci Flat, Energy-Momentum Tensor, Klein-Gordon Equation, Robertson-Walker Metric, Equation of State

1. INTRODUCTION

Kaluza-Klein theory shows how gravitation and electromagnetism can be unified by extending general relativity from four dimensions (4D) to five dimensions (5D). Kaluza-Klein theory assumes that the extra dimension is space-like and has the topology of the circle S^1. Moreover 5D metric functions are assumed to be independent of the extra dimension. Induced matter theory is a different theory. According to this theory 4D matter is a manifestation of 5D geometry. It shows how matter in 4D is described by the geometry of 5D. Induced matter theory does not restrict the topology of the extra dimension and is based on an unrestricted group of 5D coordinate transformations. Moreover the extra dimension may be space-like or time-like.

This paper is organized in the following way. In Section-2 basic idea of Kaluza-Klein theory is briefly reviewed. Section-3 provides a review of induced matter theory. Induced matter theory is a consequence of the fact that, a curved 4D Riemannian manifold can be embedded in a Ricci flat 5D Riemannian manifold. As a result 5D Einstein's equations of general relativity $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, ($\alpha$, β run from 0 to 4) with matter are included in 5D Kaluza-Klein equations $G_{AB} = 0$, (A, B run from 0 to 5) without matter. In Section-4 the idea of induced matter theory is illustrated with the help of the well known Ponce de Leon solution [1] of the vacuum 5D equations $R_{AB} = 0$. In Section-5, a new solution is derived which generalizes Ponce de Leon solution. Finally in Section-6, some concluding remarks are given.

2. KALUZA-KLEIN THEORY

Consider the 5D metric

$$dS^2 = \gamma_{AB} dx^A dx^B \tag{1}$$

Where

$$\gamma_{AB} = \begin{bmatrix} g_{\alpha\beta} - k^2\varphi^2 A_\alpha A_\beta & -k^2\varphi^2 A_\alpha \\ -k^2\varphi^2 A_\beta & -\varphi^2 \end{bmatrix}$$

Where γ_{AB} are independent of the 5th coordinate. Here the indices A, B run from 0 to 4 and α, β run from 0 to 3, [α, β denote conventional 4D indices]. For (1) 4D part of 5D Einstein's tensor G_{AB} is given by

$$\hat{G}_{\alpha\beta} = G_{\alpha\beta} + \frac{k^2\varphi^2}{2}\left[F_\lambda^\alpha F^{\beta\lambda} - \frac{1}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] - \varphi^{-1}\left[\varphi^{\alpha;\beta} - g^{\alpha\beta}\varphi^{\mu}{}_{;\mu}\right]$$

Where $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$ is the Faraday tensor $\varphi_\alpha = \frac{\partial\varphi}{\partial x^\alpha}$ and $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is the conventional 4D Einstein's tensor obtained from $\hat{G}_{\alpha\beta}$ by omitting those terms which depend on φ. 5D vacuum field equations $G_{AB} = 0$ imply $\hat{G}_{\alpha\beta} = 0$. This leads to the equations

$$G_{\alpha\beta} = -\frac{k^2\varphi^2}{2}\left[F_\lambda^\alpha F^{\beta\lambda} - \frac{1}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] + \varphi^{-1}\left[\varphi^{\alpha;\beta} - g^{\alpha\beta}\varphi^{\mu}{}_{;\mu}\right] \tag{2}$$

If $\varphi = 1$ equation (2) reduces to 4D Einstein's equations with matter

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

Where the coupling constant $k = \sqrt{16\pi}$ and

$$T_{\alpha\beta} = -\frac{k^2\varphi^2}{2}\left[F_\lambda^\alpha F^{\beta\lambda} - \frac{1}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right]$$

is the energy-momentum tensor of the electromagnetic field. Remaining field equations $G_{4\alpha} = 0$ and $G_{44} = 0$ reduce to Maxwell's equations $F_{\alpha;\lambda}^{\lambda} = 0$.

3. INDUCED MATTER THEORY

Main theme of induced matter theory is that 4D Einstein's equations of general relativity with matter

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \ (\alpha, \beta \text{ run from 0 to 3}) \tag{3}$$

are subsets of 5D Kaluza-Klein equations

$$G_{AB} = 0, \quad (\text{A, B run from 0 to 4}) \tag{4}$$

This is a consequence of a theorem by Campbell which states that an n-dimensional Riemannian manifold, curved or flat, can be embedded in an (n + 1)-dimensional Ricci flat Riemannian manifold.

To see how it works let us consider a 5D metric

$$dS^2 = e^{\nu}dt^2 - e^{\omega}(dr^2 + r^2 d\Omega^2) - e^{\mu}dl^2 \tag{5}$$

Where $x^0 = t$, $x^{1,2,3} = r,\theta,\varphi$ are the usual 4D coordinates, $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ and $x^4 = l$ is the extra fifth dimension. Let us assume that the metric coefficients ν,ω,μ are functions of t and l. Nonzero components of Einstein's tensor G_B^A are then given by

$$\left.\begin{aligned}
G_0^0 &= e^{-\nu}\left(-\frac{3\dot{\omega}^2}{4} - \frac{3\dot{\omega}\dot{\mu}}{4}\right) + e^{-\mu}\left(\frac{3\overset{**}{\omega}}{2} + \frac{3\overset{*}{\omega}{}^2}{2} - \frac{3\overset{*}{\mu}\overset{*}{\omega}}{4}\right)\\
G_4^0 &= e^{-\nu}\left(\frac{3\overset{*}{\dot{\omega}}}{2} + \frac{3\dot{\omega}\overset{*}{\omega}}{4} - \frac{3\dot{\omega}\overset{*}{\nu}}{4} - \frac{3\overset{*}{\omega}\dot{\mu}}{4}\right)\\
G_1^1 = G_2^2 = G_3^3 &= -e^{-\nu}\left(\ddot{\omega} + \frac{3\dot{\omega}^2}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{\omega}\dot{\mu}}{2} - \frac{\dot{\nu}\dot{\omega}}{2} - \frac{\dot{\nu}\dot{\mu}}{4}\right)\\
&\quad + e^{-\mu}\left(\overset{**}{\omega} + \frac{3\overset{*}{\omega}{}^2}{4} + \frac{\overset{**}{\nu}}{2} + \frac{\overset{*}{\nu}{}^2}{4} + \frac{\overset{*}{\omega}\overset{*}{\nu}}{2} - \frac{\overset{*}{\mu}\overset{*}{\omega}}{2} - \frac{\overset{*}{\nu}\overset{*}{\mu}}{2}\right)\\
G_4^4 &= -e^{-\nu}\left(\frac{3\ddot{\omega}}{2} + \frac{3\dot{\omega}^2}{2} - \frac{3\dot{\nu}\dot{\omega}}{4}\right) + e^{-\mu}\left(\frac{3\overset{*}{\omega}{}^2}{4} + \frac{3\overset{*}{\omega}\overset{*}{\nu}}{4}\right)
\end{aligned}\right\} \tag{6}$$

Where partial derivatives with respect to t are denoted by dot '.' and partial derivatives with respect to l are denoted by star '$*$'. Kaluza-Klein equations $G_B^A = 0$ give rise to the following equations

$$\left.\begin{aligned}
G_0^0 &= e^{-\nu}\left(-\frac{3\dot{\omega}^2}{4} - \frac{3\dot{\omega}\dot{\mu}}{4}\right) + e^{-\mu}\left(\frac{3\overset{**}{\omega}}{2} + \frac{3\overset{*}{\omega}{}^2}{2} - \frac{3\overset{*}{\mu}\overset{*}{\omega}}{4}\right) = 0\\
G_4^0 &= e^{-\nu}\left(\frac{3\overset{*}{\dot{\omega}}}{2} + \frac{3\dot{\omega}\overset{*}{\omega}}{4} - \frac{3\dot{\omega}\overset{*}{\nu}}{4} - \frac{3\overset{*}{\omega}\dot{\mu}}{4}\right) = 0\\
G_1^1 = G_2^2 = G_3^3 &= -e^{-\nu}\left(\ddot{\omega} + \frac{3\dot{\omega}^2}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{\omega}\dot{\mu}}{2} - \frac{\dot{\nu}\dot{\omega}}{2} - \frac{\dot{\nu}\dot{\mu}}{4}\right)\\
&\quad + e^{-\mu}\left(\overset{**}{\omega} + \frac{3\overset{*}{\omega}{}^2}{4} + \frac{\overset{**}{\nu}}{2} + \frac{\overset{*}{\nu}{}^2}{4} + \frac{\overset{*}{\omega}\overset{*}{\nu}}{2} - \frac{\overset{*}{\mu}\overset{*}{\omega}}{2} - \frac{\overset{*}{\nu}\overset{*}{\mu}}{2}\right) = 0\\
G_4^4 &= -e^{-\nu}\left(\frac{3\ddot{\omega}}{2} + \frac{3\dot{\omega}^2}{2} - \frac{3\dot{\nu}\dot{\omega}}{4}\right) + e^{-\mu}\left(\frac{3\overset{*}{\omega}{}^2}{4} + \frac{3\overset{*}{\omega}\overset{*}{\nu}}{4}\right) = 0
\end{aligned}\right\} \tag{7}$$

According to induced matter theory the new terms due to the fifth dimension in $G_0^0 = 0$ is to be identified with ρ and the new terms in $G_1^1 = 0$ with p. Collecting terms which depend on μ or derivatives with respect to l we define

$$G_0^0 = -\frac{3}{4}e^{-\nu}\dot{\omega}^2 + 8\pi\rho = 0 \tag{8}$$

$$G_1^1 = -e^{-\nu}(\ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \frac{\dot{\nu}\,\dot{\omega}}{2}) - 8\pi\, p = 0 \tag{9}$$

Where

$$8\pi\,\rho = -\frac{3}{4}e^{-\nu}\dot{\omega}\dot{\mu} + \frac{3}{2}e^{-\mu}(\overset{**}{\omega} + \overset{*}{\omega}{}^2 - \frac{\overset{*}{\mu}\,\overset{*}{\omega}}{2}) \tag{10}$$

$$8\pi\, p = e^{-\nu}(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{\omega}\dot{\mu}}{2} - \frac{\dot{\nu}\dot{\mu}}{4}) \;- e^{-\mu}(\overset{**}{\omega} + \frac{3\overset{*}{\omega}{}^2}{4} + \frac{\overset{**}{\nu}}{2} + \frac{\overset{*}{\nu}{}^2}{4} + \frac{\overset{*}{\omega}\overset{*}{\nu}}{2} - \frac{\overset{*}{\mu}\,\overset{*}{\omega}}{2} - \frac{\overset{*}{\nu}\,\overset{*}{\mu}}{4}) \tag{11}$$

Equations (8) and (9) together with $G_2^2 = 0$ and $G_3^3 = 0$ give the usual 4D Einstein's equations with matter

$$G_\beta^\alpha = 8\pi T_\beta^\alpha$$

and allow us to interpret 4D matter described by T_β^α as a manifestation of 5D geometry. The remaining equations $G_4^0 = 0$ and $G_4^4 = 0$ can be interpreted as describing some properties of matter.

We have demonstrated induced matter theory by considering a 5D diagonal metric in spatially isotropic form where the extra coordinate is space-like. If the 5D metric has the form

$$dS^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - R^2 d\Omega^2 + \varepsilon\, e^{\mu}dt^2$$

Where the metric functions $\nu, \lambda, R,\ \mu$ depend in general on t, r, l and ε = -1or 1. Then the nonzero components of energy-momentum tensor T_β^α are given by

$$8\pi T_0^0 = -e^{-\nu}\left(\frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} - \frac{\lambda'\mu'}{4} + \frac{\mu''}{2} + \frac{\mu'^2}{4}\right)$$

$$-\varepsilon\, e^{-\mu}\left(\frac{\overset{**}{\lambda}}{2} + \frac{\overset{*}{\lambda}{}^2}{4} - \frac{\overset{*}{\mu}\,\overset{*}{\lambda}}{4} + \frac{\overset{*}{R}\,\overset{*}{\lambda}}{R} - \frac{\overset{*}{R}\,\overset{*}{\mu}}{R} + \frac{\overset{*}{R}{}^2}{R^2} + \frac{2\overset{**}{R}}{R}\right)$$

$$8\pi T_0^1 = -e^{-\lambda}\left(\frac{\dot{\mu}'}{2} + \frac{\dot{\mu}\mu'}{4} - \frac{\nu'\dot{\mu}}{4} - \frac{\dot{\lambda}\mu'}{4}\right) \tag{12}$$

$$8\pi T_1^1 = -e^{-\nu}\left(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} + \frac{\nu'\mu'}{4}\right)$$

$$-\varepsilon\, e^{-\mu}\left(\frac{\overset{*}{R}{}^2}{R^2} + \frac{2\overset{**}{R}}{R} + \frac{\overset{*}{R}\,\overset{*}{\nu}}{R} - \frac{\overset{*}{R}\,\overset{*}{\mu}}{R} + \frac{\overset{**}{\nu}}{2} + \frac{\overset{*}{\nu}{}^2}{4} - \frac{\overset{*}{\nu}\,\overset{*}{\mu}}{4}\right)$$

$$8\pi T_2^2 = -e^{-\nu}\left(\frac{\dot{R}\dot{\mu}}{2R} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4}\right) + e^{-\lambda}\left(\frac{R'\mu'}{2R} + \frac{\mu''}{2} + \frac{\mu'^2}{4} - \frac{\lambda'\mu'}{4} + \frac{\nu'\mu'}{4}\right)$$

$$-\varepsilon e^{-\mu}\left(\frac{\overset{**}{R}}{R} + \frac{\overset{*}{R}\overset{*}{\nu}}{2R} + \frac{\overset{*}{R}\overset{*}{\lambda}}{2R} - \frac{\overset{*}{R}\overset{*}{\mu}}{2R} + \frac{\overset{**}{\nu}}{2} + \frac{\overset{*}{\nu}^2}{4} + \frac{\overset{**}{\lambda}}{2} + \frac{\overset{*}{\lambda}^2}{4} + \frac{\overset{*}{\nu}\overset{*}{\lambda}}{4} - \frac{\overset{*}{\nu}\overset{*}{\mu}}{4} - \frac{\overset{*}{\mu}\overset{*}{\lambda}}{4}\right)$$

$$T_3^3 = T_2^2$$

where dot '.' denotes partial differentiation with respect to t, prime '/' denotes partial differentiation with respect to r and star '$*$' denotes partial differentiation with respect to l.

In induced matter theory an important coordinate system, called matter gauge, is one in which the 5D metric is block diagonal,

$$g_{AB}(x^c) = \begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & \xi\varphi^2 \end{bmatrix}$$

Where $\xi^2 = 1$. Here ξ is a factor which allows a space-like as well as a time-like signature of the extra dimension. The procedure to obtain the 4D energy-momentum tensor T_β^α from the 5D field equations $G_{AB} = 0$ is exactly the same as above. The usual 4D Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ then hold, where

$$8\pi T_{\alpha\beta} = \frac{\varphi_{\alpha;\beta}}{\varphi} - \frac{\xi}{2\varphi^2}\left[\frac{\overset{*}{\varphi}\overset{*}{g}_{\alpha\beta}}{\varphi} - \overset{**}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}}{2}\right.$$

$$\left.+\frac{g_{\mu\nu}}{4}\{\overset{*}{g}^{\mu\nu}\overset{*}{g}_{\mu\nu} + (g^{\mu\nu}\overset{*}{g}_{\mu\nu})^2\}\right] \qquad (13)$$

Where semicolon denotes ordinary 4D covariant derivative

Equation (13) describes 4D matter $T_{\alpha\beta}$ as a manifestation of pure 5D geometry. Interpretation of the remaining equations $G_{04} = 0$ and $G_{44} = 0$ is not so straightforward. However if we let [2]

$$P_{\alpha\beta} = k(m_i u_\alpha u_\beta + m_g g_{\alpha\beta})$$

where k is a constant, m_i and m_g are the inertial and gravitational masses of a particle in the induced matter fluid and $u^\alpha = \frac{dx^\alpha}{ds}$ is its 4-velocity then these equations provide the equation $P^\beta_{\alpha;\beta} = 0$, which can be interpreted as the 4D geodesic equation and the remaining equation can be identified with the Klein- Gordon equation $\Box\varphi = m^2\varphi$.

Though close to relativity theory, neither Kaluza-Klein theory nor induced matter theory may be called relativity in 5D. Firstly because both are based on a flat 5D space and secondly neither is independent of the choice of coordinates in the five dimensional spaces.

4. EXPLICIT SOLUTIONS

Let us consider the simplest type of 5D metric

$$ds^2 = dt^2 - e^{\lambda(t)}(dr^2 + r^2 d\Omega^2) - e^{\mu(t)} dl^2 \tag{14}$$

The nonzero components of G_B^A for the metric (14) are given by

$$G_0^0 = -\left(\frac{3\dot{\lambda}^2}{4} + \frac{3\dot{\lambda}\dot{\mu}}{4}\right)$$

$$G_1^1 = G_2^2 = G_3^3 = -\left(\ddot{\lambda} + \frac{3\dot{\lambda}^2}{4} + \frac{\ddot{\mu}}{2} + \frac{3\dot{\mu}^2}{4} + \frac{\dot{\lambda}\dot{\mu}}{2}\right)$$

$$G_4^4 = \frac{3\ddot{\lambda}}{2} + \frac{3\dot{\lambda}^2}{2}$$

Identifying the terms which depend on μ or derivatives with respect to l which occur in equation $G_0^0 = 0$ with ρ and in equation $G_1^1 = 0$ with p we obtain

$$G_0^0 = -\frac{3\dot{\lambda}^2}{4} + 8\pi\rho = 0$$

$$G_1^1 = -\ddot{\lambda} + \frac{3\dot{\lambda}^2}{4} - 8\pi p = 0$$

Where $8\pi\rho = -\frac{3\dot{\lambda}\dot{\mu}}{4}$ (15)

$$8\pi p = \frac{\ddot{\mu}}{2} + \frac{3\dot{\mu}^2}{4} + \frac{\dot{\lambda}\dot{\mu}}{2} \tag{16}$$

Equation

$$G_4^4 = \frac{3\ddot{\lambda}}{2} + \frac{3\dot{\lambda}^2}{2} = 0 \tag{17}$$

provides an equation for $\dot{\lambda}$. Using equation (17) we obtain

$$\lambda = \log(at + b) \tag{18}$$

$\mu(t)$ is to be obtained by equating

$$G_0^0 = -\frac{3\dot{\lambda}}{4}(\dot{\lambda} + \dot{\mu}) = 0 \tag{19}$$

and $$G_1^1 = -\ddot{\lambda} + \frac{3\dot{\lambda}^2}{4} - 8\pi p = 0 \tag{20}$$

From (19) we obtain $\dot{\mu} = -\dot{\lambda}$. This is consistent with (20). Thus we get the solution

$\lambda = \log(at+b)$, $\mu = -\log(at+b)$.

From (15) and (16) we get the density and pressure. These are given by

$$8\pi\rho = \frac{3a^2}{4(at+b)^2},\ 8\pi p = \frac{a^2}{4(at+b)^2}$$

The equation of state given by this solution is given by $\rho = 3p$ which is typical of radiation. This is a cosmological solution.

An important class of cosmological solutions to $R_{AB} = 0$ (or $G_{AB} = 0$) is due to Ponce de Leon [1]. This is given by

$$dS^2 = l^2dt^2 - t^{\frac{2}{\alpha}} l^{\frac{2}{1-\alpha}} (dr^2 + r^2 d\Omega^2) - \alpha^2(1-\alpha)^{-2} t^2 dl^2 \tag{21}$$

Comparing with (5) we find

$$e^{\nu} = l^2,\ e^{\omega} = t^{\frac{2}{\alpha}} l^{\frac{2}{1-\alpha}},\ e^{\mu} = \alpha^2(1-\alpha)^{-2} t^2$$

Density and pressure of the cosmological fluid for the solution (21) can be obtained by putting these into the equations (10) and (11) respectively, which gives

$$8\pi\rho = \frac{3}{\alpha^2 l^2 t^2} \text{ and } 8\pi p = \frac{2\alpha - 3}{\alpha^2 l^2 t^2}$$

Equation of state is given by

$$p = \frac{2\alpha - 3}{3}\rho$$

Equation of state depends on the arbitrary constant α. For $\alpha = \frac{3}{2}$ we get

$$8\pi\rho = \frac{4}{3T^2},\ p = 0$$

where T = l t denotes the proper time. In this case the solution is identical to 4D Einstein-de Sitter model for the late universe with dust. For $\alpha = 2$ we get

$$8\pi\rho = \frac{3}{4T^2},\ 8\pi p = \frac{1}{4T^2},\ p = \frac{\rho}{3}$$

In this case the solution is identical to the 4D standard model for the early universe with radiation.

5. NEW SOLUTIONS

In this section we present a new solution which generalizes the well known Ponce de Leon solution [1]. The new solution is obtained from the featureless flat 5D metric by transformation of coordinates. In deriving the new solution we do not need to solve Kaluza-Klein field equations $R_{AB} = 0$ since it is obtained from the flat 5D metric by changing coordinates so that the equations $R_{AB} = 0$ are automatically satisfied.

Let us start with the flat 5D metric

$$dS^2 = dT^2 - dR^2 - R^2 d\Omega^2 - dL^2 \tag{22}$$

We wish to change the coordinates $(T, R, L) \rightarrow (t, r, l)$ such that with respect to the new coordinates (t, r, l) the metric g_{AB} is diagonal i.e. no cross term occurs. For this let

R = r(T+L)

Then the metric (22) transforms to

$$dS^2 = d(T+L)[d(T-L) - r^2 d(T+L) - 2r(T+L)dr] - (T+L)^2[dr^2 + r^2 d\Omega^2] \tag{23}$$

Next we let

$$T - L = r^2(T+L) + kt^a l^b$$

Then (23) reduces to

$$dS^2 = kd(T+L)d(t^a l^b) - (T+L)^2(dr^2 + r^2 d\Omega^2) \tag{24}$$

Finally we let

$$T + L = t^\alpha l^\beta$$

Then metric (24) becomes

$$dS^2 = k[a\alpha t^{\alpha+a-2} l^{\beta+b} dt^2 + b\beta t^{\alpha+a} l^{\beta+b-2} dl^2 + (b\alpha + a\beta) t^{\alpha+a-1} l^{\beta+b-1} dt dl]$$
$$- t^{2\alpha} l^{2\beta}(dr^2 + r^2 d\Omega^2) \tag{25}$$

Cross term appearing in (25) vanishes if we let

$$\beta = \alpha h,\ b = -ah \tag{26}$$

With (26) metric (25) reduces to

$$dS^2 = t^{\alpha+a-2} l^{(\alpha-a)h} dt^2 - t^{2\alpha} l^{2\alpha h}(dr^2 + r^2 d\Omega^2) - h^2 t^{\alpha+a} l^{(\alpha-a)h-2} dl^2 \tag{27}$$

Where we let $ak\alpha = 1$ by a choice of the constant k

Solution (27) generalizes Ponce de Leon solution [1]

$$dS^2 = l^2 dt^2 - t^{\frac{2}{\gamma}} l^{\frac{2}{1-\gamma}}(dr^2 + r^2 d\Omega^2) - \gamma^2(1-\gamma)^{-2} t^2 dl^2 \tag{28}$$

If we let $\alpha = \dfrac{1}{\gamma}$, $a = 2 - \dfrac{1}{\gamma}$ and $h = \gamma(1-\gamma)^{-1}$ then (27) reduces to metric (28).

PROPERTIES OF THE SOLUTION

Solution (27) represents a spherically symmetric cosmological model in general relativity. It reduces to Robertson-Walker metric with k = 0 on the hyper surfaces l = constant. Density and pressure of the cosmological fluid can be obtained by putting (27) into equations (10) and (11), which gives

$$\rho = \frac{3\alpha^2}{8\pi l^{(\alpha-a)h} t^{\alpha+a}} \quad \text{and} \quad p = \frac{a\alpha - 2\alpha^2}{8\pi t^{\alpha+a} l^{(\alpha-a)h}} \tag{29}$$

From the first of equations (29) we see that ρ is always positive and from the second of equations (29) we see that p could in principle be negative. Equation of state is given by

$$p = \left(\frac{a-2\alpha}{3\alpha}\right)\rho \tag{30}$$

It should be noted that the 5D metric (27) reduces to the 4D Robertson-Walker metric with k = 0 on the hypersurfaces l = constant if $\alpha + a = 2$. The solution depends on three constant parameters a, h and α. If a and α are so chosen that $\dfrac{a-2\alpha}{\alpha} = 1$ then the 4D part of (27) represents a radiation-dominated universe on the hyper-surfaces l = constant, while for $a = 2\alpha$ it represents a universe full of dust.

Equation (29) shows that both ρ and p depends on the extra coordinate l if $a \neq \alpha$. If $a = \alpha$ these become

$$\rho = \frac{3\alpha^2}{8\pi t^{2\alpha}}, \; p = \frac{-\alpha^2}{8\pi t^{2\alpha}} \tag{31}$$

Equations (31) show that ρ and p depend on α only. In this case the equation of state (30) becomes $\rho + 3p = 0$ i.e. gravitational density vanishes. If $h = \dfrac{\alpha+a}{\alpha-a}$ we obtain

$$\rho = \frac{3\alpha^2}{8\pi(l\,t)^{\alpha+a}}, \; p = \frac{a\alpha - 2\alpha^2}{8\pi(l\,t)^{\alpha+a}} \tag{32}$$

For $a = 2 - \alpha$ we get the results of Ponce de Leon solution.

6. CONCLUSIONS

The induced energy-momentum tensor $T_{\alpha\beta}$ for any solution of the Kaluza-Klein field equations $G_{AB} = 0$ (or equivalently $R_{AB} = 0$), depends on a choice of coordinates on the 5D space-time. So starting from a known solution one may generate new solutions by using transformations of coordinates. However the resulting solution may or may not be a realistic one. So the choice of an appropriate coordinate system on the 5D space-time is very important. In spite of this we have been able to find a realistic solution.

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